

Lattice Topological Field Theory on Non-Orientable Surfaces

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Abstract

The lattice definition of the two-dimensional topological quantum field theory [Fukuma, *et al*, Commun. Math. Phys. **161**, 157 (1994)] is generalized to arbitrary (not necessarily orientable) compact surfaces. It is shown that there is a one-to-one correspondence between real associative $*$ -algebras and the topological state sum invariants defined on such surfaces. The partition and n -point functions on all two-dimensional surfaces (connected sums of the Klein bottle or projective plane and g -tori) are defined and computed for arbitrary $*$ -algebras in general, and for the the group ring $A = \mathbb{R}[G]$ of discrete groups G , in particular.

1 Introduction

The introduction of topological field theory (TFT) by Witten [W1,W2], its axiomatization by Atiyah [A], and the novel approach of employing the TFT techniques to attack problems of topology and geometry [W1,W2,MS,DW], have motivated many authors to provide tools for rigorous construction of TFT models [TV,D,DJN,KS].

In the framework of lattice topological field theory (LTFT), a rigorous construction should inevitably start with a triangulation of the manifold under consideration. In three dimensions (resp. two dimensions) the basic observation [TV] (resp. [FHK,BP]) has been that the $6j$ -symbols of $U_q(sl(2, \mathbb{C}))$ and a large class of other algebras (resp. structure constants of associative algebras) obey the symmetries of a tetrahedron (resp. triangle) and satisfy identities which may be interpreted geometrically in terms of glued tetrahedra (resp. triangles). Associating state sums (partition functions) with a triangulation, one could show that the partition function is independent of the triangulation, i.e., it is a topological invariant.

In the basic definition of TFT [A], which is motivated by the path integral examples of Witten, and in the lattice models constructed afterwards, the orientability of the underlying manifold plays a crucial role. To the best of our knowledge, state sum models on non-orientable manifolds have not yet been constructed, even in two dimensions. The aim of the present paper is to construct, in general terms, topological state sums (partition functions)

and observables on non-orientable two-dimensional surfaces.

In our opinion, this direction of generalization of TFT deserves consideration for two reasons. The first of these is a possible relevance of topological correlation functions on non-orientable surfaces to the open string theory [GSW]. The second and a more fundamental reason is that mathematically, topological invariants are well defined for orientable as well as non-orientable manifolds, whereas the axioms of TFT [A], which are based on the path integral formulation of QFT, and the state sum models mentioned above, rely heavily on the orientability of the manifold. Therefore, it is desirable to see if one can generalize state sum invariants to also cover the non-orientable cases. Although our considerations are restricted to two dimensions, our basic idea seems to be generalizable to three dimensions as well.

The paper is organized as follows: In Sec. 2, a brief review of LTFT on orientable surfaces is presented. In Sec. 3, the definition of state sums on non-orientable surfaces is given and the generalized local (Matveev) moves are introduced accordingly. It is shown that the state sums, so defined, are invariant under these moves, provided that a set of consistency conditions are fulfilled. Thus, the state sums are sensitive only to the topological properties of the surface. In Sec. 4, it is shown how real associative $*$ -algebras provide the general solution of the consistency conditions. In Sec. 5, the observables are defined and for all $*$ -algebras the correlations on all two-dimensional surfaces are calculated. Sec. 6 is devoted to the study of a particular example

Figure 1: Colored triangle with complex valued C_{abc} 's.

Figure 2: Gluing two triangles

where the $*$ -algebra is taken to be the group ring of a discrete group.

2 Definition of LTFT on Oriented Surfaces [FHK]

Let Σ be a closed oriented surface of genus g , \mathbf{T}_g a triangulation of Σ . Then the partition function of the lattice model associated with \mathbf{T}_g is defined as follows: First, for an oriented triangle Δ in \mathbf{T}_g , one makes a coloring according to its orientation. That is, one gives a set of color indices running from 1 through N , to three edges of the triangle. One then assigns a complex number C_{abc} to a triangle with ordered color indices a, b, c (Fig. 1). Here it

Figure 3: The propagator g^{ab} and the three-point vertex in the dual diagram.

is assumed that C_{abc} is symmetric under cyclic permutations of the indices:

$$C_{abc} = C_{bca} = C_{cab} .$$

Note, however, that C_{abc} is not necessarily totally symmetric. Next, all the triangles of $\mathbf{T}_{\mathbf{g}}$ are glued by contracting their indices with a metric g^{ab} (Fig. 2). Thus one obtains a complex valued function of g^{ab} and C_{abc} for each triangulation $\mathbf{T}_{\mathbf{g}}$, and one interprets it as the partition function of the lattice model, $\mathcal{Z} = \mathcal{Z}(\mathbf{T}_{\mathbf{g}})$.

Alternatively the construction of the partition function can be done in the dual graph $\mathbf{T}_{\mathbf{g}}^*$ of $\mathbf{T}_{\mathbf{g}}$. Here one assigns C_{abc} to the vertices and g^{ab} to the links (Fig. 3). One further assumes that (g^{ab}) has its inverse (g_{ab}) ; and raises or lowers indices using these matrices. One should then choose the coefficients C_{abc} and g^{ab} such that the partition function is invariant under any local changes in the triangulation $\mathbf{T}_{\mathbf{g}}$ or in the dual diagram $\mathbf{T}_{\mathbf{g}}^*$.

A possible set of local moves which relates any two triangulations, is the two-dimensional version of the Matveev moves. These are the fusion transformation (Fig. 4) and the bubble transformation (Fig. 5). Fig. 6 demonstrates

Figure 4: Fusion transformation in $\mathbf{T}_{\mathbf{g}}^*$.

Figure 5: Bubble transformation in $\mathbf{T}_{\mathbf{g}}^*$.

Figure 6: Fusion transformation in $\mathbf{T}_{\mathbf{g}}$

an interpretation of the fusion transformation in the triangulation $\mathbf{T}_{\mathbf{g}}$. In Ref. [FHK], it is claimed that the bubble transformation can be expressed only in the dual diagram $\mathbf{T}_{\mathbf{g}}^*$. However, we would like to emphasize that it also has a clear interpretation in $\mathbf{T}_{\mathbf{g}}$. The meaning of the bubble transformation becomes clear only when one combines it with the fusion transformation. In fact, we can add a vertex to the left side of both diagrams in Fig. 5 and obtain Fig. 7. Now we perform a fusion transformation in the right hand figure to obtain Fig. 8. But this last equality is nothing but the barycentric subdivision in $\mathbf{T}_{\mathbf{g}}$ (Fig. 9).

Invariance of the partition function $\mathcal{Z}(\mathbf{T}_{\mathbf{g}})$ under the first and the second Matveev moves enforces the following constraints on the parameters C_{abc} and g^{ab} respectively.

$$C_{ab}^p C_{pc}^d = C_{bc}^p C_{ap}^d \quad (1)$$

$$g_{ab} = C_{ac}^d C_{bd}^c. \quad (2)$$

In Ref. [FHK], it is shown that every semisimple associative algebra A provides a solution of these constraints. The coefficients C_{ab}^c are identified

Figure 7: Bubble transformation applied to a vertex in \mathbf{T}_g^* .

Figure 8: Barycentric subdivision in \mathbf{T}_g^*

Figure 9: Barycentric subdivision in \mathbf{T}_g

with the structure constants of the associative algebra. In view of the definition of the structure constants in terms of a basis $\{\phi_a : a = 1, \dots, N\}$:

$$\phi_a \phi_b = C_{ab}^c \phi_c , \quad (3)$$

Eq. (1) is the expression of the associativity of the algebra, whereas Eq. (2) yields the metric g_{ab} in terms of the structure constants. Note that if we define $g_{ab} := \langle \phi_a, \phi_b \rangle$, then the cyclic symmetry of C_{abc} is expressed by

$$\langle \phi_a, \phi_b \phi_c \rangle = \langle \phi_a \phi_b, \phi_c \rangle$$

In order for g_{ab} to have an inverse, the algebra should be semisimple. One then has the following theorem [FHK]:

Theorem 1: *There is a one-to-one correspondence between the set of all LTFT's on orientable surfaces, as defined above, and the set of all semisimple associative algebras.*

Note that if one considers the regular representation of the algebra A :

$$[\pi(\phi_a)]_b^c = C_{ab}^c , \quad (4)$$

then one finds:

$$g_{ab} = \text{tr}([\pi(\phi_a)][\pi(\phi_b)]) , \quad (5)$$

$$C_{abc} = \text{tr}([\pi(\phi_a)][\pi(\phi_b)][\pi(\phi_c)]) . \quad (6)$$

The latter equations manifestly demonstrate the symmetry of g_{ab} and the cyclic symmetry of C_{abc} .

3 Generalization of LTFT to Arbitrary Compact Surfaces

Consider a closed (possibly non-orientable) surface Σ , a fixed triangulation Σ^α of Σ , and equip each triangle of Σ^α with an arbitrary orientation.¹ Denoting the number of triangles of Σ^α by F , one has 2^F possible ways of assigning orientations to the triangles. We shall call Σ^α together with such an assignment a *locally oriented triangulation* of Σ . Locally oriented triangulations corresponding to Σ^α are labeled by $\Sigma^{\alpha,k}$, $k = 1, 2, \dots, 2^F$. We shall denote the set of all $\Sigma^{\alpha,k}$'s by $\tilde{\Sigma}^\alpha$ and the set of all locally oriented triangulations of Σ by $\tilde{\Sigma}$, i.e.,

$$\begin{aligned}\tilde{\Sigma}^\alpha &:= \{\Sigma^{\alpha,k} : k = 1, \dots, 2^F\}, \\ \tilde{\Sigma} &:= \cup_\alpha \tilde{\Sigma}^\alpha.\end{aligned}$$

We shall construct the partition function as a real valued map $\mathcal{Z} : \{\Sigma^{\alpha k} : \forall \Sigma, \alpha, k\} \rightarrow \mathbb{R}$. By its topological invariance we mean that for a fixed surface Σ , this map has a constant value on $\tilde{\Sigma}$. Topologically, this means that \mathcal{Z} should be invariant under the following local moves in the space $\tilde{\Sigma}$:

A. **Flipping:** With a fixed triangulation we can change the orientation of any arbitrary triangle and thereby move in the subsets $\tilde{\Sigma}^\alpha$.

B. **Matveev Moves:** These enable us to interpolate between different

¹Note that here we use Σ^α rather than $\mathbf{T}_\mathbf{g}$ to denote a particular triangulation (indexed by α), for convenience.

Figure 10: Colored oriented triangle with real C_{abc} .

Figure 11: Gluing two oriented triangles.

subsets $\tilde{\Sigma}^\alpha$ and $\tilde{\Sigma}^\beta$ in $\tilde{\Sigma}$.

To construct the partition function, we proceed as follows: To each locally oriented triangle, carrying the color indices a, b , and c , we assign a real number C_{abc} according to the orientation of the triangle (Fig 10). Each pair of triangles with adjacent edges labeled by a and b , are glued together by means of contracting their indices using two types of matrices: g^{ab} or σ^{ab} , according to whether the orientations of the adjacent triangles are compatible or not, respectively, (Fig. 11). For brevity, we shall call two adjacent triangles with (in)compatible orientations, *(in)compatible triangles*.

Consistency of this prescription requires C_{abc} to be cyclically symmetric,

Figure 12: Oriented vertices, twisted and untwisted propagators.

and g^{ab} and σ^{ab} to be symmetric in their indices:

$$C_{abc} = C_{bca} = C_{cab} , \quad (7)$$

$$g^{ab} = g^{ba} , \quad \sigma^{ab} = \sigma^{ba} . \quad (8)$$

In the dual diagram, we associate a vertex to each triangle, a double line (propagator) to each common edge of two compatible triangles and a twisted double line (twisted propagator) to each common edge of two incompatible triangles. Thus, the numbers C_{abc} , g^{ab} , and σ^{ab} are assigned to the vertices, propagators, and twisted propagators, respectively, (Fig. 12).

Contracting all the indices, one obtains a real number which we interpret as the partition function of the lattice model based on the locally oriented triangulation $\Sigma^{\alpha, \mathbf{k}}$. The next step is to find out the conditions on C_{abc} , g^{ab} ,

Figure 13: Flipping transformation in $\mathbf{T}_{\mathbf{g}}$.

and σ^{ab} that would imply the invariance of \mathcal{Z} under flipping, i.e., $\mathcal{Z} = \mathcal{Z}(\tilde{\Sigma}^\alpha)$, and Matveev moves, i.e., $\mathcal{Z} = \mathcal{Z}(\tilde{\Sigma})$.

Consider a locally oriented triangulation $\Sigma^{\alpha, \mathbf{k}}$ and change the orientation of an arbitrary triangle in $\Sigma^{\alpha, \mathbf{k}}$ while the orientations of the rest of the triangles are kept unchanged. In this case, one of the cases depicted in Fig. 13 may happen. In view of Fig. (13), invariance of \mathcal{Z} under flipping leads to the following relations:

$$g^{aa'} g^{bb'} g^{cc'} C_{a'b'c'} = \sigma^{aa'} \sigma^{bb'} \sigma^{cc'} C_{a'c'b'} , \quad (9)$$

$$\sigma^{aa'} g^{bb'} g^{cc'} C_{a'b'c'} = g^{aa'} \sigma^{bb'} \sigma^{cc'} C_{a'c'b'} . \quad (10)$$

Next, we require invariance of \mathcal{Z} under local Matveev Moves. Consider an arbitrary pair of adjacent triangles. Without loss of generality, we as-

Figure 14: Fusion transformation in \mathbf{T}_g .

Figure 15: Barycentric subdivision in \mathbf{T}_g .

sign compatible orientation to this pair and perform the first Matveev move (Fig. 14). Invariance of \mathcal{Z} under this move yields the following relation for C_{abc} 's:

$$C_{da}{}^p C_{pb}{}^c = C_{ab}{}^p C_{dp}{}^c . \quad (11)$$

Next perform a barycentric subdivision of an arbitrary oriented triangle, Fig. 15. This yields the following relation:

$$g_{ab} = C_{ac}{}^d C_{bd}{}^c . \quad (12)$$

Note that once we have chosen the orientation of the triangles, the orientation of the new triangles obtained after affecting the Matveev moves is not

arbitrary. It is dictated by the external edges of the subdiagram where the Matveev moves take place.

Thus we have shown that the conditions Eqs. (9) – (12) are the necessary and sufficient conditions for the invariance of the partition function under the local moves in the space $\tilde{\Sigma}$. In the next section, we shall provide the general solution of these conditions.

4 General Solutions

Let A be an associative semisimple $*$ -algebra over the field of real numbers \mathbb{R} , with the $*$ -operation $\sigma : A \rightarrow A$, with $\sigma^2 = id$ and $\sigma(ab) = \sigma(b)\sigma(a)$. Further, suppose that A is equipped with an inner product $\langle \cdot, \cdot \rangle : A \times A \rightarrow \mathbb{R}$ and σ is self-adjoint with respect to this inner product.

In an arbitrary basis $\{\phi_a : a = 1, \dots, N\}$, σ is expressed by a matrix (σ_a^b) , i.e: $\hat{\phi}_a = \sigma \phi_a = \sigma_a^b \phi_b$, and the conditions on σ take the following form:

$$\sigma_a^b \sigma_b^c = \delta_a^c \quad (\text{involutiveness}) , \quad (13)$$

$$C_{ba}^c \sigma_c^{c'} = \sigma_a^{a'} \sigma_b^{b'} C_{a',b'}^{c'} \quad (\text{antihomomorphism}) , \quad (14)$$

$$\sigma_{ab} = \sigma_{ba} \quad (\text{self-adjointness}) . \quad (15)$$

Note that

$$\sigma_{ab} = \langle \phi_a, \sigma \phi_b \rangle = \langle \phi_a, \sigma_b^{b'} \phi_{b'} \rangle = \sigma_b^{b'} g_{ab'} = \sigma_{ba} ,$$

also

$$\sigma_b^a = g^{aa'} \sigma_{a'b} = g^{aa'} \sigma_{ba'} = \sigma_b^a .$$

Figure 16: Flipping transformation in $\mathbf{T_g}^*$.

One can use Eqs. (13) and (15) to write Eq.(14) in the following equivalent form

$$C_{ba}{}^c = \sigma_a{}^{a'} \sigma_b{}^{b'} \sigma_{c'}{}^c C_{a',b'}{}^{c'} . \quad (16)$$

Defining the metric as before, i.e., according to Eq. (12), we find that Eqs. (14) and (16) are precisely the necessary relations (9) and (10) for the formulation of LTFT on arbitrary (not necessarily orientable) compact surfaces. In fact, the relation with $*$ -algebras can be seen quite naturally, if one translates Fig. 13 into the dual language (Fig. 16), where a vertex shows the fusion of two elements of the algebra. In the remainder of this article, we shall use a single line, rather than a double line, to indicate a propagator and a single line with a dot to indicate a twisted propagator, for simplicity.

In view of these considerations, we have proven:

Theorem 2: *There is a one-to-one correspondence between the set of all LTFT's on two-dimensional compact surfaces (orientable or not) defined as above, and the set of all semisimple real associative $*$ -algebras.*

We conclude this section by recalling a couple of examples of associative real $*$ -algebras:

- 1) Let A be the algebra of real n -dimensional matrices $M_n(\mathbb{R})$ with the inner product $\langle a, b \rangle = \text{tr}(ab^t)$ and σ be the transpose operation $\sigma(a) = a^t$. A natural basis of $M_n(\mathbb{R})$ is provided by the matrices E_{ij} with $i, j = 1, \dots, n$ defined by $(E_{ij})_{kl} := \delta_{ik}\delta_{jl}$. We then have $\langle E_{ij}, E_{kl} \rangle = g_{ij,kl} = N\delta_{ik}\delta_{jl}$, $g^{ij,kl} = \frac{1}{N}\delta^{ik}\delta^{jl}$, and $\sigma_{ij,kl} = N\delta_{il}\delta_{jk}$.
- 2) Let $A = \mathbb{R}(G)$ be the group ring of a finite group G . For any two elements a and b of G , we define $\langle a, b \rangle = \text{tr}[\pi(a)\pi(b)]$ where π denotes the regular representation of G , and induce an inner product on $\mathbb{R}(G)$ by linear extension. We also choose the $*$ -operation to be the (linear extension of the) group inversion, $\sigma(a) := a^{-1}$. Then, it is easy to check that σ is self-adjoint:

$$\begin{aligned}
\langle a, \sigma(b) \rangle &= \text{tr}[\pi(a)\pi(b^{-1})] \\
&= [\pi(a)]_d^c [\pi(b^{-1})]_c^d \\
&= C_{ad}^c C_{b^{-1}c}^d
\end{aligned}$$

$$\begin{aligned}
&= \delta(ad, c)\delta(b^{-1}c, d) \\
&= \delta(ab^{-1}c, c) \\
&= |G|\delta_{a,b} = \langle \sigma(a), b \rangle,
\end{aligned}$$

where $\delta(a, b) := \delta_{ab}$ is the kronecker delta function, i.e.,

$$\delta(a, b) := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

5 Physical Observables and Correlation Functions

Let Σ be a (compact and connected) surface with an n -component boundary. The boundary of Σ is homeomorphic to the union of n disjoint circles. Although Σ itself may not be orientable, each component of its boundary may be oriented. Let us assign the color indices a_1, a_2, \dots, a_n to the n circles comprising the boundary. We denote such a surface and a locally oriented triangulation of it by Σ_{a_1, \dots, a_n} and $\Sigma_{a_1, \dots, a_n}^{\alpha, k}$, respectively. We shall define the partition function, $\mathcal{Z}(\Sigma_{a_1, \dots, a_n}^{\alpha, k})$, such that it will be completely independent of the triangulation and will depend only on the color indices and the orientations of the boundary components. For definition of the partition function we use exactly the same set of rules as for the closed surfaces plus the following:

Every boundary element with index a , whose orientation is (in)compatible with that of the triangle adjacent to it, corresponds to a (twisted) untwisted external line in the dual diagram (Fig. 17). Two different sur-

Figure 17: Triangle adjacent to a boundary component.

)

faces are glued along their common boundary when the orientations of the boundaries are opposite.

We define the insertion of the operators O_a ($a = 1, 2, \dots, N$) into the correlation functions as creating a loop boundary with a fixed color index a and summing over all other color indices of the triangulation. We denote the correlation functions of O_{a_1}, \dots, O_{a_n} on a closed surface Σ by $\langle O_{a_1} \cdots O_{a_n} \rangle_\Sigma$. Next, we prove:

Theorem 3: *The value of $\mathcal{Z}(\Sigma_{a_1, \dots, a_n}^{\alpha, k})$ is independent of the triangulation, i.e., $\mathcal{Z} = \mathcal{Z}(\Sigma_{a_1, \dots, a_n})$.*

Proof: We should only take care of the triangles adjacent to the external lines. Consider a flipping in the triangle adjacent to a boundary component (Fig. 18). In the dual diagram this flipping is demonstrated also by Fig. 16. We know that due to Eqs. (9) and (10), the partition function is invariant under such moves. In Fig. 18, we may also consider other possibilities for the orientations of the boundary components and

Figure 18: Flipping a triangle adjacent to a boundary component.

the triangles, and see that the invariance of the correlation functions imposes no extra conditions besides Eqs. (9) and (10).

Note however that the correlation functions are invariant under a reversal of the orientation of all the boundary components. This marks a \mathbb{Z}_2 -symmetry of our construction. In particular, this implies that the one-point functions do not depend on the orientation of the boundary. This is due to the fact that although one can compare two different orientations of a given boundary component, one cannot compare the orientations of two different boundary components. Thus, it is impossible to assign an intrinsic value (\pm) to a given orientation. This then means that for a fixed set of indices on the n boundary components of Σ , one can define 2^{n-1} different correlation functions. In the next section, we shall see how one can obtain all these 2^{n-1} different correlation functions from the knowledge of only one of them.

In the remainder of this section, we present some explicit calculations.

Calculation of Correlation Functions

In the following we pursue the calculation, in general terms and without specifying the underlying algebra, of the following quantities:

The partition function of

A - the sphere,

B - the projective plane,

C - the Klein bottle,

the one-point functions on

D - the sphere,

E - the projective plane,

F - the Klein bottle,

and finally,

G - the two-point function on the sphere,

H - the three-point function on the sphere, and

I - the partition and correlation functions on arbitrary compact surfaces.

We shall see that observables D, E, F, and H can be used as building blocks for calculation of all correlation functions on arbitrary compact surfaces, i.e.,

Figure 19: Graphical identities

I. In our graphical calculations, we shall use the identities depicted in Fig. 19.

Next, we pursue the computation of:

A- Partition function of the sphere S^2

We can always normalize the partition function of the sphere to unity. For future use we present in Fig. 20, the simplest triangulation of the 2-sphere together with its dual graph.² By performing second Matveev move in the

²Note that the multiple arrows on the edges of some of triangles are used to mean that they are to be identified. They are not to be confused with the single arrows which specify

Figure 20: A triangulation of the sphere and its dual diagram.

dual graph, we see that the dual diagram of S^2 is a circle. Therefore we have:

$$\mathcal{Z}(S^2) = \mathcal{Z}(\bigoplus) = \mathcal{Z}(\bigcirc) = 1 .$$

B-Partition function of the projective plane $\mathbb{R}P^2$

A simple triangulation of the projective plane and the corresponding dual graph is shown in Fig. 21. In order to compute the partion function, first we simplify the dual diagram by performing the first and then the second Matveev moves in the lower area. The result is demonstrated in Fig. 22. From the latter diagram we obtain:

$$\mathcal{Z}(\mathbb{R}P^2) = C_{ca}{}^b C_{db}{}^a \sigma^{cd} . \tag{17}$$

the orientations of the boundary components.

Figure 21: A triangulation of $\mathbb{R}P^2$ and its dual graph.

Figure 22: A simplified dual diagram for $\mathbb{R}P^2$.

Figure 23: A triangulation of the Klein bottle and its dual diagram.

C- Partition function of the Klein bottle \mathcal{K}

Fig. 23 shows a triangulation of the Klein bottle and its dual diagram, where we have also indicated how to simplify the dual diagram using Matveev moves. In view of Fig. 23, we obtain :

$$\mathcal{Z}(\mathcal{K}) = C_{b'c'}^a C_{cba} \sigma^{cc'} \sigma^{bb'} \quad (18)$$

D- One-point function on the sphere (disk)

Removing the interior of a circle from the sphere and fixing an index a on the circle (Fig. 24), we obtain the one-point function on the sphere , which is topologically a disk. Hence, we have

$$\langle O_a \rangle_{S^2} = C_{ab}^b . \quad (19)$$

Figure 24: A triangulation of the disk.

Figure 25: A triangulation of the Mobius strip.

E- One-point function on the projective plane (Mobius strip)

The simplest triangulation of the one-point function on the projective plane is shown in Fig. 25. This is obtained by removing the interior of a circle from $\mathbb{R}P^2$. Topologically, this corresponds to the Mobius strip. In view of Fig. 25, we have:

$$\langle O_a \rangle_{\mathbb{R}P^2} = C_{abc} \sigma^{bc} . \quad (20)$$

Figure 26: A triangulation of the one-point function on Klein bottle.

F- One-point function on the Klein bottle

In order to compute the one-point function on the Klein bottle, we cut a disk in Fig. 23, and obtain Fig. 26. The latter leads to:

$$\langle O_a \rangle_{\mathcal{K}} = C_{ab}{}^c C_{cd'}{}^{e'} C_{ed}{}^b \sigma^{dd'} \sigma_{e'}{}^e . \quad (21)$$

G- Two-point functions on the sphere

According to the orientations of the boundaries there are two different two-point functions on the sphere, depicted in Fig. 27 which we call η_{ab} and ξ_{ab} . One can find the simplest triangulation of η_{ab} and ξ_{ab} by representing both of them as rectangles with two identified sides. According to Fig. 27:

$$\eta_{ab} = C_{ac}{}^d C_{db}{}^c , \quad (22)$$

$$\xi_{ab} = \eta_a{}^{b'} \sigma_{b'b} . \quad (23)$$

Gluing two η 's or two ξ 's, one can verify the following identities:

$$\eta_a{}^b \eta_b{}^c = \eta_a{}^c , \quad \eta_a{}^b \xi_b{}^c = \xi_a{}^b \eta_b{}^c = \xi_a{}^c , \quad \xi_a{}^b \xi_b{}^c = \eta_a{}^c . \quad (24)$$

Figure 27: A triangulation of the two-point functions on S^2 .

In fact, the first identity is the same as in the orientable case. The remaining two identities are consequences of Eq. (23). The significance of Eqs. (24) will be emphasized below.

In Ref. [FHK], it is shown that η is a projection onto the center $Z(A)$ of the algebra A , i.e., $\eta_a{}^b C_{bcd} = \eta_a{}^b C_{bdc}$, which implies:

$$\begin{aligned} \forall \phi \in A : \quad \eta \phi &\in Z(A) , \\ \forall \tilde{\phi} \in Z(A) : \quad \eta \tilde{\phi} &= \tilde{\phi} . \end{aligned} \tag{25}$$

Moreover, in view of Eq. (23), ξ also acts as a projector to the center $Z(A)$, although it is not a proper projection due to the last relation in (24).

Note that by gluing ξ_{ab} to any boundary component of the surface, we can change its prescribed orientation. Thus the correlation functions correspond-

ing to different assignments of the orientation to the boundary components may be obtained in this way from a given one.

At this stage, we would like to relabel the indices of the basis $\{\phi_a : a = 1, \dots, N\}$ of A in such a way that the first M indices label the basis of $Z(A)$:

$$A = \bigoplus_{a=1}^N \mathbb{R}\phi_a = Z(A) \oplus Z^c(A) := \left(\bigoplus_{\alpha=1}^M \mathbb{R}\phi_\alpha \right) \oplus \left(\bigoplus_{i=M+1}^N \mathbb{R}\phi_i \right). \quad (26)$$

Since $\eta = (\eta_a^b)$ is a projector onto $Z(A)$ and Eq. (25) holds, η takes the following form in the new basis:

$$\begin{aligned} (\eta_{ab}) &= \begin{bmatrix} \eta_{\alpha\beta} = g_{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix}, \\ (\eta_a^b) &= \begin{bmatrix} \eta_\alpha^\beta = \delta_\alpha^\beta & 0 \\ 0 & 0 \end{bmatrix}, \\ (\eta^{ab}) &= \begin{bmatrix} \eta^{\alpha\beta} = g^{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (27)$$

An interesting observation is that σ induces a \mathbb{Z}_2 -grading of the center $Z(A)$, although it does not induce such a grading on the whole algebra A . Thus, we have:

$$Z(A) = Z^+(A) \oplus Z^-(A) = \left(\bigoplus_{\alpha^+=1}^{M_1} \mathbb{R}\phi_{\alpha^+} \right) \oplus \left(\bigoplus_{\alpha^-=M_1+1}^M \mathbb{R}\phi_{\alpha^-} \right), \quad (28)$$

where $\sigma\phi_{\alpha^\pm} = \pm\phi_{\alpha^\pm}$, and

$$\begin{aligned} Z^+(A) Z^+(A) &\subset Z^+(A) \\ Z^+(A) Z^-(A) &\subset Z^-(A) \\ Z^-(A) Z^-(A) &\subset Z^-(A). \end{aligned} \quad (29)$$

Figure 28: A triangulation of a three-point function on S^2 with a prescribed orientation on the boundary components.

H - Three-point functions on the sphere

The simplest triangulation for the three-point function on sphere, with the prescribed orientations as shown in Fig. 28, leads to a dual diagram consisting of three η 's joint at a vertex [FHK]. Thus, we have:

$$N_{abc} := \langle O_a O_b O_c \rangle = \eta_a^{a'} \eta_b^{b'} \eta_c^{c'} C_{a'b'c'} . \quad (30)$$

Note that in view of Eqs. (27),

$$N_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} . \quad (31)$$

Other choices of orientations on the boundary components correspond to replacing some of η 's by ξ 's in Eq. (30).

Since every insertion of operator O_a (to obtain a multi-point function) is necessarily subject to the projection by η or ξ , the following theorem [FHK] also generalizes to the case considered in this paper.

Theorem 4: *The set of physical observables is in one-to-one correspondence with the center $Z(A)$ of the the real associative $*$ -algebra A associated with the LTFT. In particular, the number of the independent physical operators is equal to the dimension of $Z(A)$.*

In view of Eq. (31) and the \mathbb{Z}_2 -grading of $Z(A)$ demonstrated by Eqs. (29), we can regard $O_{\alpha+}$ and $O_{\alpha-}$ as “bosonic” and “fermionic” observables. This terminology is motivated by the following “selection rules”:

$$N_{\alpha+\beta+\gamma-} = N_{\alpha-\beta-\gamma-} = 0 .$$

I - Case of general compact surfaces

To compute the correlation functions of other compact surfaces, we appeal to the following result:

Theorem 5: *The one-point functions on the sphere D_α , the Klein bottle K_α , the projective plane \mathcal{M}_α , and the three-point function on sphere $N_{\alpha\beta\gamma}$ can be used as building blocks to find any correlation function on any compact connected surface by gluing.*

Proof: First note that by gluing a disk D_α to a three-point function $N_{\alpha\beta\gamma}$ on the sphere, one obtains the two-point function $\eta_{\alpha\beta}$ on the

sphere. Gluing $\eta_{\alpha\beta}$ to $N_{\alpha\beta\gamma}$, one obtains a handle operator which is used in the construction of surfaces of higher genus. Furthermore, gluing $N_{\alpha\beta\gamma}$ to any n -point function yields an $(n+1)$ -point function on the same surface. Next, one can glue \mathcal{M}_α (resp. \mathcal{K}_α) to the $(n+1)$ -point function on a genus g orientable surface Σ_g to obtain the n -point function on the non-orientable surface $\Sigma_g \# \mathbb{R}P^2$ (resp. $\Sigma_g \# \mathcal{K}$). According to the classification theorem for two-dimensional surfaces [M], this exhausts all the possibilities of the multi-point functions on arbitrary compact surfaces.

These considerations can be expressed in an algebraic language by defining the matrices:

$$(N_\beta)_\alpha^\gamma := N_{\alpha\beta}^\gamma ,$$

the vectors ω , \mathcal{M} , and \mathcal{K} with components:

$$\omega_\alpha := \text{tr}(N_\alpha) , \quad \mathcal{M}_\alpha , \quad \mathcal{K}_\alpha ,$$

respectively, and the matrix:

$$\tilde{N} := \sum_{\alpha=1}^M \omega_\alpha N_\alpha .$$

Denoting by g the genus of the surface, we will then have for the orientable surfaces Σ_g :

$$\langle O_{\alpha_1} \cdots O_{\alpha_n} \rangle_{g=0} = \left(N_{\alpha_2} N_{\alpha_3} \cdots N_{\alpha_{n-1}} \right)_{\alpha_1}^{\alpha_n} , \quad (32)$$

$$\langle O_{\alpha_1} \cdots O_{\alpha_n} \rangle_{g=1} = \text{tr} (N_{\alpha_1} N_{\alpha_2} \cdots N_{\alpha_n}) , \quad (33)$$

$$\begin{aligned}
\langle O_{\alpha_0} \rangle_g &= (N_{\alpha_0} N_{\alpha_1} \cdots N_{\alpha_g}) \omega_{\alpha_1} \omega_{\alpha_2} \cdots \omega_{\alpha_g} , \\
&= \left(\tilde{N}^{g-1} \omega \right)_{\alpha_0} ,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\langle O_{\alpha_1} \cdots O_{\alpha_n} \rangle_g &= \langle O_{\alpha_1} \cdots O_{\alpha_n} O_{\alpha_{n+1}} \rangle_{g=0} \langle O_{\alpha_{n+1}} \rangle_g , \\
&= \left(N_{\alpha_2} \cdots N_{\alpha_n} \tilde{N}^{g-1} \omega \right)_{\alpha_1} ,
\end{aligned} \tag{35}$$

$$\mathcal{Z}(\Sigma_g) = \omega_{\alpha_1} \cdots \omega_{\alpha_g} \langle O_{\alpha_1} \cdots O_{\alpha_g} \rangle_{g=0} = \omega^t \tilde{N}^{g-2} \omega , \tag{36}$$

and for non-orientable surfaces:

$$\mathcal{Z}(\Sigma_g \# \mathcal{K}) = \mathcal{K}_\alpha \langle O_\alpha \rangle_g = \mathcal{K}^t \tilde{N}^{g-1} \omega , \tag{37}$$

$$\mathcal{Z}(\Sigma_g \# \mathbb{R}P^2) = \mathcal{M}_\alpha \langle O_\alpha \rangle_g = \mathcal{M}^t \tilde{N}^{g-1} \omega , \tag{38}$$

$$\langle O_\alpha \rangle_{\Sigma_g \# \mathcal{K}} = (\mathcal{K}^t \tilde{N}^g)_\alpha , \tag{39}$$

$$\langle O_\alpha \rangle_{\Sigma_g \# \mathbb{R}P^2} = (\mathcal{M}^t \tilde{N}^g)_\alpha , \tag{40}$$

$$\begin{aligned}
\langle O_{\alpha_1} \cdots O_{\alpha_n} \rangle_{\Sigma_g \# \mathcal{K}} &= \langle O_{\alpha_1} \cdots O_{\alpha_{n+1}} \rangle_{g=0} \langle O_{\alpha_{n+1}} \rangle_{\Sigma_g \# \mathcal{K}} , \\
&= \left(N_{\alpha_2} \cdots N_{\alpha_n} \tilde{N}^g \mathcal{K} \right)_{\alpha_1} ,
\end{aligned} \tag{41}$$

$$\langle O_{\alpha_1} \cdots O_{\alpha_n} \rangle_{\Sigma_g \# \mathbb{R}P^2} = \left(N_{\alpha_2} \cdots N_{\alpha_n} \tilde{N}^g \mathcal{M} \right)_{\alpha_1} , \tag{42}$$

where the superscript “ t ” stands for the “transpose”.

6 Example: The Group Ring $A = \mathbb{R}(G)$

In this section we deal with the special case where $A = \mathbb{R}[G] := \bigoplus_{a \in G} \mathbb{R}a$, is a group ring associated with a finite group G of order $|G|$. In this case, one has:

$$C_{ab}^c = \delta(ab, c) . \tag{43}$$

The group ring A is naturally a real $*$ -algebra with the $*$ -operation given by linear extention of:

$$\sigma(a) := a^{-1} , \quad \forall a \in G . \quad (44)$$

Using Eqs. (43) and (44), we have:

$$g_{ab} = |G| \delta(a, b^{-1}) , \quad (45)$$

$$C_{abc} = |G| \delta(abc, 1) , \quad (46)$$

$$\sigma_{ab} = |G| \delta(a, b) . \quad (47)$$

Similarly, we find

$$g^{ab} = \frac{1}{|G|} \delta(a, b^{-1}) , \quad (48)$$

$$\sigma^{ab} = \frac{1}{|G|} \delta(a, b) , \quad (49)$$

$$\sigma_b^a = \sigma_b^a = \delta(a, b^{-1}) , \quad (50)$$

In view of these equations, we may easily compute:

$$\eta_{ab} = \langle O_a O_b \rangle_0 = \frac{|G|}{h_{[a]}} \delta([a], [b^{-1}]) . \quad (51)$$

Here, $[a]$ denotes the conjugacy class of a , i.e.,

$$[a] := \{b \in G : b = g a g^{-1} , g \in G\} ,$$

and $h_{[a]}$ is the number of elements of $[a]$. Furthermore, we have

$$\eta_a^b = \eta_{ac} g^{cb} = \frac{1}{h_{[a]}} \delta([a], [b]) , \quad (52)$$

$$\xi_{ab} = \eta_a^c \sigma_{cb} = \frac{|G|}{h_{[a]}} \delta([a], [b]) , \quad (53)$$

$$\xi_a^b = \xi_{ac} g^{cb} = \frac{1}{h_{[a]}} \delta([a], [b^{-1}]) . \quad (54)$$

Next, we consider some specific examples:

1. The partition function for the sphere S^2 :

$$\mathcal{Z}(S^2) = \mathcal{Z}(\bigoplus) = C_{abc} C_{a'b'c'} g^{aa'} g^{bb'} g^{cc'} = \frac{1}{|G|} \sum_{a,b,c} \delta(abc, 1) \delta(a^{-1}c^{-1}b^{-1}, 1) = 1 . \quad (55)$$

2. One-point function on S^2 (The disk (D)):

$$\langle O_a \rangle_{S^2} = C_{ab}^b = \sum_b \delta(ab, b) = |G| \delta(a, 1) . \quad (56)$$

3. The partition function for the projective plane $\mathbb{R}P^2$:

$$\begin{aligned} \mathcal{Z}(\mathbb{R}P^2) &= C_{ab}^c C_{dc}^b \sigma^{da} \\ &= \frac{1}{|G|} \sum_{a,b,c,d} \delta(ab, c) \delta(dc, b) \delta(d, a) \\ &= \frac{1}{|G|} \sum_a \delta(a^2, 1) . \end{aligned}$$

The sum in the latter equation can be split into a sum over the distinct conjugacy classes $[b]$, followed by a sum over the elements belonging to each class, $a \in [b]$. Then, in view of the identity:

$$\sum_{a \in [b]} \delta(a^2, 1) = \frac{|G|}{h_{[b^2]}} \delta([b^2], 1) ,$$

one finally has:

$$\mathcal{Z}(\mathbb{R}P^2) = \sum_{[b]} \frac{1}{h_{[b^2]}} \delta([b^2], 1) . \quad (57)$$

4. One-point function on $\mathbb{R}P^2$ (the Mobius strip (\mathcal{M})):

$$\begin{aligned}
\langle O_a \rangle_{\mathbb{R}P^2} &= \mathcal{M}_a = C_{abc} \sigma^{cb} = C_{ab}{}^c \sigma_c{}^b \\
&= \sum_{b,c} \delta(ab, c) \delta(c, b^{-1}) \\
&= \sum_b \delta(ab, b^{-1}) =: G_a^{1/2} .
\end{aligned} \tag{58}$$

Here, $G_a^{1/2}$ is the number of elements of G whose square equals a . Note that $G_a^{1/2}$ is a function of $[a]$. To see this suppose that $b_i, i = 1, \dots, G_a^{1/2}$ are such that $b_i^2 = a$. Then for all $g \in G$, $b'_i := g b_i g^{-1}$ have the property that $b_i'^2 = g a g^{-1} = a' \in [a]$. Thus, $G_{gag^{-1}}^{1/2} = G_a^{1/2}$.

5. The partition function of the Klein bottle (\mathcal{K}):

$$\mathcal{Z}(\mathcal{K}) = C_{b'c'}{}^a C_{cba} \sigma^{cc'} \sigma^{bb'} = \sum_{[a]} \frac{1}{h_{[a]}} \delta([a], [a^{-1}]) . \tag{59}$$

6. One-point function on \mathcal{K} :

$$\langle O_a \rangle_{\mathcal{K}} = C_{ab}{}^m C_{md'}{}^{e'} C_{ed}{}^b \sigma^{dd'} \sigma_e{}^{e'} = \sum_{[b]} \frac{1}{h_{[b]}} \delta([ab], [b^{-1}]) . \tag{60}$$

We conclude this section emphasizing the fact that all the correlation functions are functions of the conjugacy classes. This is to be expected since the physical observables are related to the center of the algebra and the center is spanned by the conjugacy classes. Furthermore, the physical observables being functions only of the conjugacy classes can be expressed in terms of the characters of the irreducible representations of the group.

7 Conclusion

In this article, it is shown how in two dimensions one can formulate state sums on non-orientable compact manifolds. Pursuing the same approach as in the treatment of the orientable case, one encounters the problem of the lack of a canonical orientation for the non-orientable surfaces. This manifests itself in the lack of a canonical prescription for the assignment of ordered C_{abc} 's to the triangles of a given triangulation. The solution offered above involves the following three steps:

- 1) Introduction of locally oriented triangulations,
- 2) Generalization of the Matveev moves, i.e., inclusion of flipping transformation.
- 3) Employing the $*$ -structure of real associative $*$ -algebras to ensure the topological invariance of the partition and correlation functions.

Thus, at a more fundamental level, the \mathbb{Z}_2 -obstruction of non-orientability leads to the requirement of the existence of a $*$ -structure for the underlying algebra of any LTFT on non-orientable manifolds.

A similar problem exists in three dimensions where adjacent tetrahedra with incompatible orientations are present in any triangulation. It seems that our approach may be applied to this case, as well.

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